ENS 2013/2014
Exam of General Relativity, november the 14^{th} 2013
9h30-12h30

In all the exam, except if mentioned otherwise, one takes $c = 1$. The symbol "$\equiv$" indicates a definition. One uses the conventions of the lectures. Answers should be carefully justified.

1. Weyl's tensor, conformally equivalent and conformally flat metrics, Nordström's scalar theory.

One considers a space-time of dimension D with $D \geq 3$. This space-time is endowed (like in general relativity) with a metric (a two times covariant symmetric tensor) of signature $-, +, +, +, \cdots$. One assumes that when $D \neq 4$ the definitions given in the lectures for the tensors, covariant derivatives (etc...) still hold. We remind in particular the definitions of Christoffel's symbols $\Gamma^\mu_{\nu\rho}$, Riemann tensor $R^\lambda{}_{\mu\nu\rho}$, Ricci tensor $R_{\mu\nu}$, and Ricci scalar R:

$$\begin{aligned} \Gamma^\mu_{\nu\rho} &= \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right) && (1) \\ R^\lambda{}_{\mu\nu\rho} &= \partial_\nu \Gamma^\lambda_{\mu\rho} - \partial_\rho \Gamma^\lambda_{\mu\nu} + \Gamma^\sigma_{\mu\rho} \Gamma^\lambda_{\sigma\nu} - \Gamma^\sigma_{\mu\nu} \Gamma^\lambda_{\rho\sigma} && (2) \\ R_{\mu\nu} &= R^\sigma{}_{\mu\sigma\nu} && (3) \\ R &= R^{\mu\nu} g_{\mu\nu} && (4) \end{aligned}$$

1.1. Do the symmetries (not involving derivatives) of the Riemann tensor valid for $D = 4$ still hold when $D \neq 4$?

1.2. We remind that an event P of space-time being chosen, one can build a coordinate system, called a locally inertial coordinate system in P, such that, in P, the first derivatives of the metric vanish. What is then the expressions of the components of the Riemann tensor in P in the locally inertial coordinate system?

1.3. The Christoffel symbols allow one to define a covariant derivative for arbitrary D like the one defined for $D = 4$. We will denote ∇ this covariant derivative, as in the lectures. Show that for an arbitrary tensor $T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}$, one has

$$\begin{aligned} \left[\nabla_\rho, \nabla_\lambda \right] T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} &\equiv \nabla_\rho \nabla_\lambda T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} - \nabla_\lambda \nabla_\rho T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} && (5) \\ &= -\sum_{i=1}^{i=p} R_{\rho\lambda\sigma}{}^{\mu_i} T^{\mu_1 \cdots \mu_{i-1} \sigma \mu_{i+1} \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} + \sum_{j=1}^{j=q} R_{\rho\lambda\nu_j}{}^\sigma T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_{j-1} \sigma \nu_{j+1} \cdots \nu_q}. && (6) \end{aligned}$$

Indication: use locally inertial coordinates (one should then carefully justify how one can deduce the expression above from the one obtained using locally inertial coordinates, as well as the different steps of the reasoning).

1.4. The covariant derivative satisfies the Jacobi identities given by

$$\left([\nabla_\mu, [\nabla_\nu, \nabla_\rho]] + [\nabla_\nu, [\nabla_\rho, \nabla_\mu]] + [\nabla_\rho, [\nabla_\mu, \nabla_\nu]] \right) T = 0, \quad (7)$$

where T denotes here a tensor of arbitrary variance and $[,]$ denotes a commutator as defined in (5). Show that one can deduce

$$\nabla_\sigma R^\lambda{}_{\mu\nu\rho} + \nabla_\nu R^\lambda{}_{\mu\rho\sigma} + \nabla_\rho R^\lambda{}_{\mu\sigma\nu} = 0 \quad (8)$$

1.5. One defines then the 4-times covariant tensor

$$C_{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho} + a \left(g_{\lambda\nu} R_{\mu\rho} + g_{\mu\rho} R_{\lambda\nu} - g_{\lambda\rho} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\rho} \right) + b \left(g_{\lambda\nu} g_{\mu\rho} - g_{\lambda\rho} g_{\mu\nu} \right) R \quad (9)$$

where a and b are real numbers. Show that the tensor $C_{\lambda\mu\nu\rho}$ has the same symmetries (not involving derivatives) as the Riemann tensor.

1.6. Show that one can choose a and b depending on the dimension D such that all contractions of any two indices of $C_{\lambda\mu\nu\rho}$ with the inverse metric $g^{\sigma\omega}$ vanish. In this case, $C_{\lambda\mu\nu\rho}$ is called the Weyl tensor. Give the corresponding expressions of a and b as functions of D.

1.7. One says that two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ are conformally equivalent, if and only if there exists a coordinate system x^μ and a scalar function $S(x^\mu)$ such that one has

$$g_{\mu\nu} = S \times f_{\mu\nu} \tag{10}$$

What can be said on the light cones of two conformally equivalent metrics?

1.8. One says that a space-time is conformally flat, if there exists a coordinate system where its metrics $g_{\mu\nu}$ is conformally equivalent to the canonical metric of Minkowski space-time $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1, \cdots)$. In such a coordinate system, one has then

$$g_{\mu\nu} = S \times \eta_{\mu\nu} \tag{11}$$

and S is called the conformal factor of the metric. One considers FLRW space-times with flat spatial sections, i.e. space-times with a line element ds given by

$$ds^2 = -dt^2 + a^2(t) dx^i dx^j \delta_{ij}, \tag{12}$$

$a(t)$ being the scale factor and δ_{ij} the canonical metric of the $D-1$ dimensional euclidian space. Show that such a FLRW space-time is conformally flat. What is the expression of its conformal factor?

1.9. One considers a conformally flat space-time and one denotes by $S = e^{2\varphi}$ the conformal factor. Show that the Christoffel symbols for this metric read

$$\Gamma^\lambda_{\mu\nu} = \delta^\lambda_\nu \partial_\mu \varphi + \delta^\lambda_\mu \partial_\nu \varphi - \eta_{\mu\nu} \eta^{\lambda\sigma} \partial_\sigma \varphi \tag{13}$$

Show that one can expect the Ricci tensor to have the form

$$R_{\mu\nu} = c_1 \partial_\mu \partial_\nu \varphi + c_2 \partial_\mu \varphi \partial_\nu \varphi + c_3 \eta_{\mu\nu} \eta^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + c_4 \eta_{\mu\nu} \eta^{\rho\sigma} \partial_\rho \partial_\sigma \varphi \tag{14}$$

where coefficients c_1, c_2, c_3 and c_4 are constants. Can one expect that these constants be independent of the dimension? One assumes that φ is vanishing at an event P. Give a sufficient condition on the derivatives $\partial_\mu \varphi$ such that the coordinate system is locally inertial in P.

1.10. One considers now (in questions 1.10., 1.11. and 1.12.) a theory which differs from general relativity and which has been discussed by Nordström (in 1912) and then Einstein-Fokker (1914). In this theory (that we will call here Nordström scalar theory) space-time is, like in general relativity, endowed with a metric $g_{\mu\nu}$. One assumes besides that $D = 4$ for this theory. Matter is also described by a conserved energy momentum tensor $T^{\mu\nu}$ which obeys then $\nabla_\mu T^{\mu\nu} = 0$. The metric is supposed to be conformally flat, i.e.

$$g_{\mu\nu} = \Omega^2(x) \eta_{\mu\nu}, \tag{15}$$

and Einstein equations are replaced by

$$R = kT \tag{16}$$

where k is a constant and $T \equiv g^{\mu\nu} T_{\mu\nu}$. One considers a set of dust particles, i.e. a pressureless fluid described by $T^{\mu\nu} = \rho u^\mu u^\nu$, where ρ is the mass density and $u^\mu \equiv dx^\mu / d\tau$ (τ being the proper time along the trajectory of the particle) the unit velocity 4-vector ($u^\mu u_\mu = -1$). Show that like in general relativity the conservation of $T^{\mu\nu}$ implies non only the conservation of matter $\nabla_\mu (\rho u^\mu) = 0$, but also the geodesic equation $u^\mu \nabla_\mu u^\nu = 0$.

1.11. Do the weak equivalence principle hold in Nordström scalar theory (i.e. do a gold bar and an apple fall in the same way in a gravitational field)?

1.12. One admits that for a metric of the form (15) the Ricci scalar reads (in 4 dimensions)

$$R = -6\Omega^{-3}\eta^{\mu\nu}\partial_\mu\partial_\nu\Omega \tag{17}$$

One assumes that in the weak field limit, $\Omega(x^\lambda) = 1 - U(x^\lambda) + \cdots$, where one neglects higher order irrelevant terms and where $U \ll 1$. What should be the form of U in order to find back (at lowest order) the newtonian motion in a gravitational field of a static point source described by the metric considered here? Is it possible to choose k so that the Newtonian theory is recovered in this limit? Does Nordström scalar theory pass all the tests of General Relativity?

(**NB: The following two questions 1.13. and 1.14. lead to quite long calculations. It is advised to look at them only at the end, if you have some time left**).

1.13. Show that the Weyl tensor of a conformally flat metric is vanishing identically (whatever the dimension $D \geq 3$) and verify the expression (17).

1.14. Show that the Weyl tensor is vanishing in dimension 3.

2. de Sitter space-time

2.1. One considers a 4 dimensional surface embedded in a 5 dimensional Minkowski space-time. This surface is characterized by coordinates $X^A(x^\mu)$, with $\mu = 0,1,2,3$ et $A = 0,1,2,3,4$, X^A being canonical coordinates on Minkowski space-time, and x^μ coordinates on this surface. Show that the flat metric η_{AB}, of the embedding space-time, induces a metric (called the "induced metric") $g_{\mu\nu}$ on the surface, which reads

$$g_{\mu\nu} = \partial_\mu X^A \partial_\nu X^B \eta_{AB}. \tag{18}$$

Indication: consider the way one can compute "distances" (in the Riemannian sense) on the surface.

2.2. One considers, in the 5 dimensional Minkowski space-time of line element ds given by

$$\begin{aligned} ds^2 &= \eta_{AB} dX^A dX^B & (19)\\ &= -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2, & (20)\end{aligned}$$

the surface defined by

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = H^{-2}, \tag{21}$$

H being a constant, i.e. de Sitter space-time. One sets

$$\begin{aligned} X^i &= e^{Ht}x^i \quad \text{for} \quad i = 1,2,3, \\ X^0 - X^4 &= 2e^{Ht}. & (22)\end{aligned}$$

Show that the induced metric on de Sitter space-time is of Friedmann-Lemaître-Robertson-Walker (FLRW) form

$$ds^2 = -dt^2 + a(t)^2 \eta_{ij} dx^i dx^j \tag{23}$$

(with $x^0 = t$) , and compute $a(t)$.

2.3. Do the above coordinates cover all the de Sitter space-time?

3. Cosmological readshift for massive particles

One considers a particle of non vanishing mass in geodesic motion in a FLRW metric of the form

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu & (24)\\ &= -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \\ &= -dt^2 + a^2(t)\gamma_{ij} dx^i dx^j & (25)\end{aligned}$$

where we remind that $a(t)$ is the "scale factor" of the Universe. One has $k = 0, +1, -1$ respectively for "flat" universes (euclidian spatial sections), "closed" universes (spatial sections given by 3-spheres) or "open" universes (hyperbolic spatial sections), and γ_{ij} is the corresponding spatial metric. The latter describes a maximally symmetric space of signature $+, +, +$.

3.1. Are the x^i necessarily constant for such a particle? Remind why one can choose the proper time τ of the particle as an affine parameter for the geodesic motion.

3.2. Show that the geodesic motion leads to the equation

$$\frac{d^2x^0}{d\tau^2} + \frac{\dot{a}}{a}|\vec{u}|^2 = 0 \tag{26}$$

where $\vec{u} = d\vec{x}/d\tau$ is the 3-velocity of the particle and $|\vec{u}|^2 = a^2\gamma_{ij}u^iu^j$ is its square norm. If one defines the 4-velocity as $u^\mu = dx^\mu/d\tau$ one has then $u^\mu = (u^0, \vec{u})$

3.3. What relation does exist between u^0 and $|\vec{u}|$?

3.4. Deduce from above that $\frac{|\dot{\vec{u}}|}{|\vec{u}|} = -\frac{\dot{a}}{a}$ where a dot means a derivative with respect to the time t.

3.5. Show then that if $\vec{p}$ denotes the 3-momentum of the particle, $\vec{p}$ does redshift with the cosmic expansion. Is the found relation the same as the one valid for photons?

3.6. What is the kinetic energy loss for a massive particle in geodesic motion as measured by an observer following the particle (i.e. constantly in a frame where the observer is at rest with respect to the particle) between the cosmological redshift (denoted by z in the lectures) corresponding to the emission of the CMB until that corresponding to the formation of the first stars?

4. Palatini identities and Einstein-Hilbert action

One considers a 4 dimensional space-time endowed with a metric $g_{\mu\nu}$. Let $\delta g_{\mu\nu}$, be a variation of the metric $g_{\mu\nu}$ (with $\delta g_{\mu\nu} \ll g_{\mu\nu}$). I.e. one replaces $g_{\mu\nu}$ by $g_{\mu\nu} + \delta g_{\mu\nu}$ (that we denote by $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$) and in the following one studies the induced change (always denoted similarly with a δ) on various quantities when we do the replacement $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$.

4.1. What is, at lowest order, the variation $\delta g^{\mu\nu}$ of the inverse metric as function of $\delta g_{\mu\nu}$?

4.2. Show that at the linearized order in $\delta g_{\mu\nu}$ the variation $\delta\Gamma^\lambda_{\mu\nu}$ of $\Gamma^\lambda_{\mu\nu}$, is given by

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left\{\nabla_\nu(\delta g_{\rho\mu}) + \nabla_\mu(\delta g_{\rho\nu}) - \nabla_\rho(\delta g_{\mu\nu})\right\} \tag{27}$$

Is $\delta\Gamma^\lambda_{\mu\nu}$ a tensor?

4.3. Show that the variation $\delta R_{\mu\nu}$ of $R_{\mu\nu}$ is given by the Palatini identity:

$$\delta R_{\mu\nu} = \nabla_\lambda\left(\delta\Gamma^\lambda_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^\lambda_{\mu\lambda}\right) \tag{28}$$

4.4. Show then that $\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}$ reads

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \partial_\lambda\left(\sqrt{-g}A^\lambda\right), \tag{29}$$

where A^λ is a vector that should be determined.

4.5. Deduce that the variation of $\sqrt{-g}R$ is given by

$$\delta\left(\sqrt{-g}R\right) = -\sqrt{-g}G^{\mu\nu}\delta g_{\mu\nu} + \partial_\lambda\left(\sqrt{-g}A^\lambda\right) \tag{30}$$

4.6. Deduce the field equations (Euler-Lagrange equations) for the metric, derived from the Einstein-Hilbert action S_{EH} given by

$$S_{EH} = \int d^4x\sqrt{-g}R. \tag{31}$$