TEL AVIV UNIVERSITY אוניברסיטת תל-אביב
RAYMOND AND BEVERLY SACKLER FACULTY OF EXACT SCIENCES
SCHOOL OF PHYSICS & ASTRONOMY
הפקולטה למדעים מדוייקים
ע"ש ריימונד ובברלי סאקלר
בית הספר לפיסיקה ואסטרונומיה

The exam is three hours long.
No support materials are allowed at the exam.

Instructions: Answer the first question (30 points) and choose 2 of the next 3 (35 points each). Explain each step briefly (e.g., avoid writing a series of mathematical formulas with no explanation).

Formulas that you may use without explanation if you find them helpful:

$$k(R,t) = \frac{\partial f(R,t)}{\partial R} \qquad (m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma|k| + k^2 c_s^2$$

$$V_c^2 = \frac{GM(R)}{R} \qquad \beta = \theta - \alpha \qquad \gamma = \frac{4GM(\xi)}{\xi c^2} \qquad \mu = \left|\frac{\theta d\theta}{\beta d\beta}\right|$$

$$\Sigma_1(R,\phi,t) = H(R,t)e^{i[m\phi + f(R,t)]} \qquad \frac{\omega}{m} = \Omega \pm \frac{\kappa}{m}$$

$$\hat{\delta}(\vec{k}) \equiv \int \frac{d^3x}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}}\delta(\vec{x}) \qquad \sigma^2 = \int d^3k\, \tilde{W}(\vec{k})\tilde{W}(-\vec{k})P(\vec{k})$$

$$\delta(\vec{x}) = \int d^3k\, e^{i\vec{k}\cdot\vec{x}}\,\hat{\delta}(\vec{k}) \qquad \kappa^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2$$

$$\tilde{W}_{TH}(\vec{k}) = \frac{3}{(kR)^3}\left[\sin(kR) - kR\cos(kR)\right]$$

$$\delta^{\mathrm{D}}(\vec{k}) = \int \frac{d^3x}{(2\pi)^3} e^{\pm i\vec{k}\cdot\vec{x}} \qquad \xi(r) = 4\pi\int_0^\infty k^2 dk\; P(k)\,\frac{\sin(kr)}{kr}$$

Question 1: Required

In this question, write simple equations or approximate expressions (Include clear explanations, but no need for full mathematical derivations).

a. In spiral structure, what is the winding problem? What is the tight-winding approximation?
[10 points]
The winding problem is the fact that if a spiral arm is made up of fixed material, then it will be wound many times over the lifetime of the galaxy. The number of windings is around:
(Omega t)/(2 pi) = (v_c t)/(2 pi R) = (200 km/s) 10^10 yr /(2 pi 8 kpc) = 40. Real spiral arms have only a few windings. Conclusion: a spiral arm is a pattern, and does not contain fixed material.
The tight-winding approximation is the assumption that the radial wavelength of the spiral solution is small compared to the radius, or equivalently that the pitch angle is very small (i.e., the spiral arm is nearly tangential). This approximation is required for a simple analytical solution of spiral structure.

b. What is a singular isothermal sphere? Why is it important in astrophysics, and what objects does it describe well?
[10 points]
An SIS is a simple spherical solution to the collisionless Boltzmann equation, with a distribution function assumed to be spherically symmetric, steady-state, and dependent only on energy. Its three-dimensional density profile is $1/r^2$. This solution for collisionless particles is mathematically identical to an isothermal gas solution. It is a good description of the mass distribution of galactic halos and thus used to model gravitational lensing.

c. What is the Press-Schechter model? Describe its assumptions and results.
[10 points]
This is a simple model for the number density of dark matter halos as a function of mass. Its assumptions are a Gaussian random field of density fluctuations, linear growth of perturbations, and spherical collapse. Its results are a distribution dn/dM (the "halo mass function") that depends on the ratio of δ_c (the critical collapse threshold) to $\sigma(M)$ (the root-mean-square fluctuation on a scale M).

Question 2: Spiral arms

Consider spiral structure within a thin gas disk with a surface density: $\Sigma \propto R^{-n}$, where R is the two-dimensional radial distance in the disk. Assume that 0<n<2 . You may assume the tight-winding approximation.

a. Find Ω(R) and κ(R).
[10 points]

$\Sigma = \Sigma_0 (R/R_0)^{-n}$ so: $M = \int 2\pi R dR \, \Sigma = \frac{2\pi}{2-n} \Sigma_0 R_0^{\,2} (R/R_0)^{2-n}$

Also, $\Omega = \frac{V_c}{R} = \sqrt{\frac{GM}{R^3}} = \sqrt{\frac{2\pi G \Sigma_0}{2-n}} R_0^{\,n/2} R^{-(n+1)/2}$ and

$\kappa^2 = \Omega^2[-(n+1)+4]$ so $\kappa = \Omega\sqrt{3-n}$.

b. Find the radii of the Lindblad resonances and of the corotation resonance, for a given wave solution with temporal frequency ω and m spiral arms.
[10 points]

Let $\Omega_p = \omega / m$, then corotation: $\Omega = \Omega_p$, so radius is:

$$R_{cor} = \left(\frac{\Omega_p}{R_0^{\,n/2}} \sqrt{\frac{2-n}{2\pi G \Sigma_0}} \right)^{-2/(n+1)}$$

Lindblad: $\Omega_p = \Omega \pm \frac{\kappa}{m} = \Omega \left[1 \pm \frac{\sqrt{3-n}}{m} \right]$ so radii are:

$$R_{\pm} = R_{cor} \left(1 \pm \frac{\sqrt{3-n}}{m} \right)^{2/(n+1)}$$

c. Find the dispersion relation for this disk, at R>0, assuming ω « Ω(R) and that the sound speed c_S « R*Ω(R). What values of m can be described with your solution? Find the equation (in polar coordinates) describing the shape of a spiral arm for the case m=1.
[10 points]

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma|k| + k^2c_s^2$$

Neglecting the ω and c_S terms, we get:

$|k| = \frac{(3-n-m^2)\Omega^2}{2\pi G\Sigma} = \frac{3-n-m^2}{2-n}\frac{1}{R}$. Now, m^2 must be no more than 3-n, and 0<n<2, so m=0 or 1.

m=1: k=1/R, $f = \int RdR = \ln(R)$. From the phase of the spiral solution, the shape of a spiral arm is: $\phi = \phi_0 - \ln(R/R_0)$.

d. Analyze the stability of this disk to axisymmetric perturbations. (Here, do **not** assume ω « Ω or c_S « R*Ω(R) .) What is the condition for having more stability at large radii than at small radii?

[5 points]

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma|k| + k^2c_s^2$$

Now set m=0:

$\omega^2 = \kappa^2 - 2\pi G\Sigma|k| + k^2{c_s}^2$, want $\omega^2 > 0$ for all |k| so:

$$1 < Q = \frac{c_s\kappa}{\pi G\Sigma} = \frac{c_s\Omega\sqrt{3-n}}{\pi G\Sigma} \propto R^{(n-1)/2}$$

Thus, more stability at large R means n>1 .

Question 3: The power spectrum and correlation function

a. Write down a broken power-law approximation (i.e., including two different power-law segments) for the shape of the density power spectrum in the universe at present. Explain (roughly, not in full quantitative detail) what (in terms of cosmic history) sets the peak position (i.e., wavenumber) k_{peak} .
[10 points]

$P(k)=A(k/k_{peak})$ for $k< k_{peak}$, $A(k/k_{peak})^{-3}$ for $k>k_{peak}$.The peak position is set by the horizon at matter-radiation equality.

b. Write down a full expression for the correlation function corresponding to the power spectrum from part a. Also, find the same for the specific distance: $r=1/k_{peak}$. (No need to evaluate any integrals)
[10 points]

$$\xi(r) = 4\pi \int_0^{k_{peak}} k^2 dk\, A \frac{k}{k_{peak}} \frac{\sin(kr)}{kr} + 4\pi \int_{k_{peak}}^{\infty} k^2 dk\, A \left(\frac{k}{k_{peak}}\right)^{-3} \frac{\sin(kr)}{kr}$$

$$= \frac{4\pi A}{k_{peak} r^4} \int_0^{k_{peak} r} \sin(x) x^2 dx + 4\pi A k_{peak}^{\ 3} \int_{k_{peak} r}^{\infty} \frac{\sin(x)}{x^2} dx$$

$$\xi(1/k_{peak}) = 4\pi A k_{peak}^{\ 3} \left[\int_0^1 \sin(x) x^2 dx + \int_1^{\infty} \frac{\sin(x)}{x^2} dx \right]$$

c. Evaluate the following expectation value:
$\left\langle \hat{\delta}(\vec{k}_1) \hat{\delta}(\vec{k}_2) \right\rangle$, where $\hat{\delta}$ is the Fourier transform of the density perturbation. Show that the result can be written in the form:
$P(k_1)\delta^{\mathrm{D}}(\vec{k}_1 + \vec{k}_2)$ (in terms of a Dirac Delta function), and find $P(k_1)$ in terms of the correlation function.
Hint: Once you obtain an expression that includes spatial position variables $\vec{x}_1, \vec{x}_2$, change variables to their mean and their difference.
[15 points]

$$\left\langle \hat{\delta}(\vec{k}_1)\hat{\delta}(\vec{k}_2)\right\rangle = \left\langle \int \frac{d^3x_1}{(2\pi)^3} e^{-i\vec{k}_1\cdot\vec{x}_1}\delta(\vec{x}_1)\int \frac{d^3x_2}{(2\pi)^3} e^{-i\vec{k}_2\cdot\vec{x}_2}\delta(\vec{x}_2)\right\rangle$$

$= \iint \frac{d^3x_1}{(2\pi)^3}\frac{d^3x_2}{(2\pi)^3} e^{-i(\vec{k}_1\cdot\vec{x}_1+\vec{k}_2\cdot\vec{x}_2)}\xi(r)$, where r is the distance between $\vec{x}_1,\vec{x}_2$. We now change variables to: $\vec{r} = \vec{x}_1 - \vec{x}_2 \quad , \quad \vec{R} = (\vec{x}_1 + \vec{x}_2)/2$

Check absolute value of the change of variables determinant: In each dimension we have: $\left|\det\begin{pmatrix} 1 & -1 \\ 1/2 & 1/2 \end{pmatrix}\right| = 1$. Also,

$\vec{x}_1 = \vec{R} + \frac{1}{2}\vec{r} \quad , \quad \vec{x}_2 = \vec{R} - \frac{1}{2}\vec{r}$ Thus we get:

$\iint \frac{d^3r}{(2\pi)^3}\frac{d^3R}{(2\pi)^3} e^{-i\left[\vec{R}\cdot(\vec{k}_1+\vec{k}_2)+\vec{r}\cdot(\vec{k}_1-\vec{k}_2)/2\right]}\xi(r)$. The R integral gives a Dirac Delta function:

$\delta_D(\vec{k}_1 + \vec{k}_2)\int \frac{d^3r}{(2\pi)^3} e^{-i\vec{r}\cdot\vec{k}_1}\xi(r)$, where we set $\vec{k}_2 = -\vec{k}_1$ inside the remaining integral. So:

$$P(k_1) = \int \frac{d^3r}{(2\pi)^3} e^{-i\vec{r}\cdot\vec{k}_1}\xi(r)$$

Question 4: Gravitational lensing

Assume a lensing mass distribution with a surface density: $\Sigma \propto \xi^{-n}$, where ξ is the projected distance in the lens plane. Assume that 0<n<2 .

a. Find the Einstein angle in this case, and then write the lens equation where the source and lens positions are expressed in units of the Einstein angle.
[15 points]

$\Sigma = \Sigma_0 (\xi / \xi_0)^{-n}$ so $M = \int 2\pi\xi d\xi \, \Sigma = \dfrac{2\pi}{2-n} \Sigma_0 \xi_0{}^2 (\xi/\xi_0)^{2-n}$ and the deflection angle (with units c=1) is $\dfrac{4GM}{\xi} = \dfrac{8\pi G}{2-n} \Sigma_0 \xi_0 (\xi/\xi_0)^{1-n}$.

Plug in $\xi = D_L \theta$, and then the lens equation is:

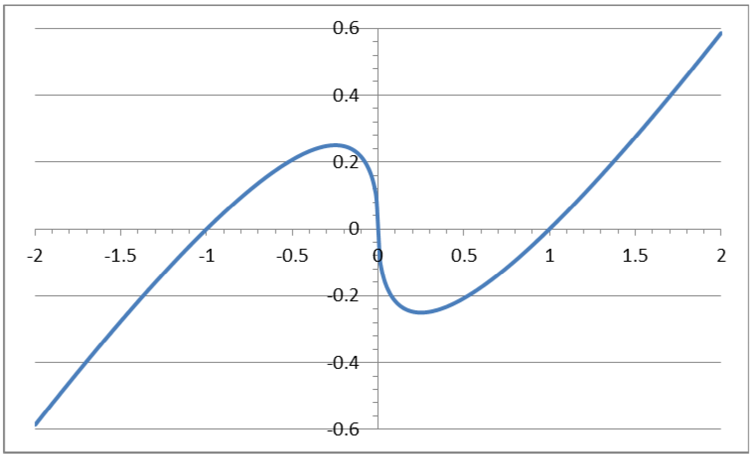
$\beta = \theta - \dfrac{D_{LS}}{D_S D_L} \dfrac{4GM}{\theta} \equiv \theta - F\theta^{1-n}$. The Einstein angle is when

$\beta = 0$ so: $\theta_E = F^{1/n} = \left(\dfrac{8\pi G \Sigma_0}{2-n} \dfrac{D_{LS}}{D_S} \right)^{1/n} \xi_0 D_L{}^{(1-n)/n}$. The lens equation is then: $u = v - v^{1-n}$ where $u = \beta / \theta_E$, $v = \theta / \theta_E$.

b. Choose n=1/2. Make a plot of the lens equation (Your plot does not need to be exactly accurate, you only need to get the right general shape and the main features). Use the plot to find the **number** of images as a function of the source position.
[10 points]

$u = v - \sqrt{v}$. When v is negative, the deflection has the same magnitude but is positive (towards the center of the lens). Plot u(v) [see next page].
Easy to see that u=0 at v=-1, 0, and 1. Also, find the local min of u at v>0 (and symmetric local max at v<0):

$\dfrac{du}{dv} = 1 - \dfrac{1}{2\sqrt{v}} = 0$, so $v = \pm \dfrac{1}{4}$, $u = \mp \dfrac{1}{4}$. For a given u, find number of solutions v:
If u>1/4 or u<-1/4, have one image. If -1/4 < u < 1/4, have three images.



c. Find the expression for the magnification of images, for the lens from part b. Find the caustics (i.e., source positions for which the magnification is infinite). What is the shape (projected on the sky) of the caustics?

[10 points]

$$\mu = \left|\frac{udu}{vdv}\right|^{-1} = \left|\frac{v}{\left(\sqrt{v}-1\right)\left(\sqrt{v}-\frac{1}{2}\right)}\right|$$

This is the answer at v>0, same (by symmetry) at v<0 if we plug in |v|.

Caustics are when v=±1 => u=0 , or v=±1/4 => $u = \mp\frac{1}{4}$, but actually (by circular symmetry) this is a whole ring at |u|=1/4. So in total, the shape of the caustics is a ring plus a point.