TEL AVIV UNIVERSITY אוניברסיטת תל אביב

Cosmology 2 Exam **SOLUTION**

Semester B 2022/3, Exam A

Lecturer: Prof. Rennan Barkana **Date:** Feb. 7, 2023

Length: 3 hours **No outside material allowed**

Instructions: Answer the first question (required: 60 points) and choose 2 of the next 3 (20 points each). In your answers, explain each step briefly.

Here are various formulas that you may use without explanation if you find them helpful (not all are relevant):

$$V_c^2 = \frac{GM(R)}{R} \qquad \beta = \theta - \alpha \qquad \gamma = \frac{4GM(\xi)}{\xi c^2} \qquad \mu = \left|\frac{\theta d\theta}{\beta d\beta}\right|$$

$$\hat{\delta}\left(\vec{k}\right) = \int d^3x\, \delta(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \; ; \qquad \delta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \hat{\delta}\left(\vec{k}\right) e^{i\vec{k}\cdot\vec{x}}$$

$$\bar{\xi}(r) = \frac{1}{2\pi^2}\int_0^\infty k^2 dk\, \tilde{W}^2(k) P(k) \frac{\sin(kr)}{kr}$$

$$\tilde{\mathrm{W}}_{\mathrm{Tophat}}(k) = \frac{3}{(kR)^3}\left[\sin(\mathrm{kR}) - kR\cos(\mathrm{kR})\right]$$

$$H^2 = \frac{8\pi G}{3}\rho \qquad \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\vec{x}}{dt}\cdot\frac{\partial}{\partial \vec{x}} f + \frac{d\vec{v}}{dt}\cdot\frac{\partial}{\partial \vec{v}} f = 0$$

Question 1: Required

In this question, write clear explanations, and try to include simple equations or approximate quantitative expressions. There is, however, **no** need for long mathematical derivations or precise numerical coefficients.

Please choose 3 out of 4:

a. What is the accepted explanation of spiral structure? Describe briefly the basic physics involved.
[20 points]

Spiral structure is a density wave pattern in the disk, which does not consist of fixed stars within the spiral density peaks. The physics that describes this is a self-consistent model in which the spiral density pattern gives rise to a gravitational potential perturbation, which in turn changes the orbits of stars, and these stellar orbits together must be consistent with the density pattern itself.

b. In spherical collapse, explain what the linearly extrapolated perturbation at collapse (equal to 1.686) is. How is it defined, and why is it useful?
[20 points]

The linearly extrapolated perturbation at collapse is the result of the linear growth solution to spherical collapse, extrapolated beyond the regime where the linear solution is valid. Specially, it is the linearly extrapolated value at which the actual, fully non-linear solution, corresponds to collapse (i.e., the spherical shell reaches a zero radius). This value is useful for approximately predicting the distribution of halos that form given an initial perturbation field. The perturbations can be easily linearly extrapolated, and then the collapse threshold can be applied in order to predict where non-linear collapse will occur. A statistical application of this idea is used in the Press-Schechter model.

c. What is matter-radiation equality? Describe its role in the evolution of density perturbations.
[20 points]

Matter-radiation equality is the cosmic moment at which the matter density equals the radiation density:

$$\rho_m / \rho_{crit,0} = \Omega_m a^{-3} = \rho_r / \rho_{crit,0} = \Omega_r a^{-4}$$

and so:

$$a = \Omega_r / \Omega_m$$

This moment is important in the evolution of density perturbations since perturbations inside the horizon grow differently before and after equality. In the radiation dominated era, there is essentially no growth (logarithmic at best), while there is steady growth in the matter dominated era. Thus, perturbations that enter the horizon prior to equality grow differently from those that enter after equality, which causes a break in the power spectrum at a scale corresponding to the size of the horizon at matter-radiation equality.

d. What is the Jeans mass, and what is the Toomre instability criterion in disks? Give a rough estimate for each of them (with a simple formula).
[20 points]

The Jeans mass is the mass of a region for which gravity and pressure are balanced. Regions with mass above the Jeans mass can gravitationally collapse. To estimate it, we compare the dynamical timescale $1/\sqrt{G\rho}$ with the time L/c_s for a pressure (sound) wave to cross the size L of the region. Then the mass is (dropping small numerical factors):

$$L^3 \rho = \left(c_s / \sqrt{G\rho}\right)^3 \rho = \left(c_s / \sqrt{G}\right)^3 / \sqrt{\rho}$$

The Toomre instability criterion gives the condition for gravitational instability/collapse, for a region within a rotating disk. To estimate it, we compare the circular velocity V_c with c_s . Stability means that the circular velocity is smaller, so:

$$1 < c_s / V_c$$

To write this in a more standard form, we use $\kappa \sim \Omega \sim V_c / R$. Also,

$$V_c \sim \sqrt{GM/R} \sim \sqrt{G\Sigma R^2 / R} \sim \sqrt{G\Sigma R}$$

Putting this together, we get:

$$1 < c_s V_c / V_c{}^2 \sim c_s (\mathrm{R}\,\kappa) / (G\Sigma \mathrm{R}) = c_s \kappa / (G\Sigma)$$

Question 2: The virial theorem and Press-Schechter

a. What is the virial theorem? When is it valid (in rough, general terms)? Illustrate it specifically for the Earth orbiting the Sun (assume a circular orbit).
[10 points]

The virial theorem states that the total kinetic energy K equals minus ½ the total potential energy U. It is valid for a steady-state, self-gravitating system.

For the Earth orbiting the sun, $\frac{V_c^2}{R} = \frac{GM}{R^2}$ and therefore:

$$\mathrm{K} = \frac{1}{2} m V_c^2 = -\frac{1}{2}\left(-\frac{GMm}{R}\right) = -\frac{1}{2}\mathrm{U}$$

b. How is the virial theorem used in spherical collapse to relate the virial radius of a halo to the radius of maximum expansion during the collapse? Explain briefly.
[5 points]

At maximum expansion (turnaround), the energy is purely gravitational, so the energy (per unit mass) of a shell containing the halo mass M is:

$$E = K + U = U = -\frac{GM}{R_t}$$

By conservation of energy, this is the same as the energy at virialization, for which we use the virial theorem:

$$E = K + U = \frac{1}{2} U = -\frac{1}{2}\frac{GM}{R_\mathrm{v}}$$

Thus, we find that $R_\mathrm{v} = R_t/2$.

c. In the Press-Schechter model, there is a missing factor of two in the derivation, which is simply multiplied by at the end. Explain briefly why this factor is missing.
[5 points]

In this model, the initial perturbations are linearly extrapolated to later times, and regions above the critical collapse threshold are assumed to have formed halos. Due to the growth of fluctuations, all positive fluctuations eventually satisfy this. Due to the initial Gaussian fluctuations, exactly half of the volume has positive fluctuations, hence the missing factor of 2.

A deeper understanding/explanation of this is the cloud-in-cloud problem: even a region that is below the threshold can be part of a larger region that does collapse into a halo. This can be shown to account for the factor of 2.

Question 3: The power spectrum and correlation function

Assume a power-law power spectrum, $P(k)\propto k^n$, and a density field that grows with time in proportion to the growing mode in an Einstein de Sitter universe.

In this question you only need to find how things scale. There is no need to find normalizations.

a. Find how the root mean square (r.m.s.) fluctuation σ in spheres of radius R scales with R and with redshift z.
[10 points]

We use the formula for correlation function:

$$\sigma^2 \sim \int_0^\infty k^2 dk\, \widetilde{W}^2(kR)P(k)$$

where k and R are comoving. Now we let $x = kR$ and get:

$$\sigma^2 \sim R^{-3-n} \int_0^\infty \mathrm{x}^2 dx\, \widetilde{W}^2(\mathrm{x}) x^n$$

And thus:

$$\sigma \sim R^{-(3+n)/2}$$

For the redshift dependence, note that $P(k) \sim {D_+}^2 \sim a^2 \sim (1+z)^{-2}$ where we used the growing mode D_+ in the Einstein de Sitter case. So:

$$\sigma \sim R^{-(3+n)/2}/(1+z)$$

b. Find how the correlation function scales with distance r and with redshift z.
[5 points]

Similarly:

$$\xi(r) \sim \int_0^\infty k^2 dk \frac{\sin(kr)}{kr} P(k)$$

so the answer comes out the same:

$$\xi \sim r^{-(3+n)/2}/(1+z)$$

c. Calculate how the typical virial temperature of collapsing halos scales with redshift. (The virial temperature is the temperature to which gas is heated when it virializes within a dark matter halo. You may use results that you remember from class about virialization.)

[5 points]

The virial temperature is defined so that the thermal energy is of order the kinetic energy, which (by the virial theorem) is of order the potential energy. Doing all of this per particle:

$k_B T_v \sim \mu V_c^2 \sim \frac{GM}{\mathrm{r}} \sim G\rho \mathrm{r}^2 \sim G\rho R^2/(1+z)^2$,

where r is the physical radius, R the comoving radius, and ρ is the physical density. Now, the virial density is a constant times the cosmic density, which goes as $(1+z)^3$ (we are in the EdS case, as stated at the beginning of this question). So:

$$T_v \propto (1+z)R^2$$

The typical collapsing halo is that for which σ is around the linear threshold for collapse. Setting $\sigma \sim 1$ in the answer to part (a) gives:

$$R \sim (1+z)^{-2/(3+n)}$$

Thus:

$$T_v \propto (1+z)^{1-(4/(3+n))} = (1+z)^{(n-1)/(3+n)}$$

Question 4: Gravitational lensing

Assume the thin lens approximation. We will assume a lensing mass that consists of two parts (R is the projected radius):

1. A point mass at R=0.
2. A distribution with projected mass density $\Sigma(R) \propto \frac{1}{R}$ as a function of R.

a. Find the projected mass within projected radius R, and write down the lens equation.
[10 points]

The point mass M_P gives a constant projected mass (at any R>0). The mass distribution gives:
$M(R) = C\int_0^R 2\pi R dR \frac{1}{R} = 2\pi C R$, where C is a constant.
The lens equation is then:

$$\beta = \theta - \alpha = \theta - \frac{D_{LS}}{D_S}\gamma$$

where $\gamma = \frac{4G}{c^2}\left(\frac{M_P}{R} + 2\pi C\right)$ and $R = D_L\theta$.
We will write the lens equation as:

$$\beta = \theta - \frac{A}{\theta} - B\,\mathrm{sign}(\theta)$$

where A and B are constants. Note that the deflection is always towards the center position (R=0), hence the need for the sign (+1 or -1) function (This was shown in an example in the class lecture on lensing).

b. Find the positions of all images. Make sure to cover all possible cases.
[10 points]

We solve:

$$0 = \theta^2 - [B\,\mathrm{sign}(\theta) + \beta]\,\theta - A$$

So:

$$\theta = \frac{1}{2}\left\{[B\,\mathrm{sign}(\theta) + \beta] \pm \sqrt{[B\,\mathrm{sign}(\theta) + \beta]^2 + 4A}\right\}$$

Since the square root is larger in magnitude than the $[B\,\mathrm{sign}(\theta) + \beta]$ term, one solution is always positive and one is always negative. Thus, there are always two solutions, and we can simplify them:

$$\theta_+ = \frac{1}{2}\left\{[B+\beta] + \sqrt{[B+\beta]^2 + 4A}\right\}$$
$$\theta_- = \frac{1}{2}\left\{[\beta-B] - \sqrt{[\beta-B]^2 + 4A}\right\}$$

There is also one separate case, which is the $\beta = 0$ (Einstein radius) case. In this case there is axial (cylindrical) symmetry, and the image is a full ring, of radius:

$$\theta_{\mathrm{E}} = \frac{1}{2}\left\{B + \sqrt{B^2 + 4A}\right\}$$